denote the width and height of each red rectangle. Also denote by L the length of the initial square. We claim that either holds:

$$\sum_{i=1}^{n} a_i \geq L \text{ or } \sum_{j=1}^{m} d_j > L.$$

Indeed, suppose that there exists a horizontal line across the square that is covered entirely with blue rectangles. Then, the total width of these rectangles is at least L, and the claim is proven. Otherwise, there is a red rectangle intersecting every horizontal one, and hence the total height of these rectangles is at least L.

Now, WLOG we can assume that $\sum_{i=1}^{n} a_i \geq L$. Applying Cauchy's inequality to vectors

$$\vec{u} = \left(\sqrt{\frac{a_1}{b_1}}, \sqrt{\frac{a_2}{b_2}}, \cdots \sqrt{\frac{a_n}{b_n}}\right)$$

and

$$\vec{v} = \left(\sqrt{(a_1 b_1}, \sqrt{a_2 b_2}, \cdots \sqrt{a_n b_n}\right),$$

$$\left(\sum_{i=1}^{n} \frac{a_i}{b_i}\right)\left(\sum_{i=1}^{n} a_i b_i\right) \geq \left(\sum_{i=1}^{n} a_i\right)^2 \geq L^2.$$

Since we know that$= \sum_{i=1}^{n} a_i b_i = \dfrac{2L^2}{3}$, then $\sum_{i=1}^{n} \dfrac{a_i}{b_i} \geq \dfrac{3}{2}$. Moreover, each $c_j \leq L$, so

$$\sum_{j=1}^{m} \frac{d_j}{c_j} = \sum_{j=1}^{m} \frac{c_j d_j}{c_j^2} \geq \frac{1}{L^2} \sum_{j=1}^{m} c_j d_j = \frac{1}{3}.$$

Adding up the preceding, yields

$$S \geq \frac{3}{2} + \frac{1}{2} = \frac{11}{6}.$$

Equality holds when $S = \dfrac{11}{6}$. It can be achieved by making the top $\dfrac{2}{3}$ of the square a blue rectangle, and the remaining $\dfrac{1}{3}$ bottom rectangle red.

- **5630:** *Proposed by Arkady Alt, San Jose, CA*

Find the integer part of the minimal value of $k + \dfrac{n}{k}, k \in N$.

Solution 1 by Michel Bataille, Rouen, France

Let μ denote the minimal value of $k + \dfrac{n}{k}$ when $k \in N$.
If $n = 0$, we clearly have $\lfloor \mu \rfloor = \mu = 1$.

Let f_n be the function defined on $(0, \infty)$ by $f_n(x) = x + \dfrac{n}{x}$.

If $n < 0$, f_n is increasing on $(0, \infty)$, hence $\mu = 1 + \dfrac{n}{1} = n + 1$ and $\lfloor \mu \rfloor = \lfloor n + 1 \rfloor$.

From now on, we suppose that $n > 0$ and for simplicity, we set $m = \lfloor \sqrt{n} \rfloor$. Note that $m^2 \leq n < (m+1)^2 = m^2 + 2m + 1$.

We prove that $\lfloor \mu \rfloor = 2m$ if $n < m^2 + m$ and $\lfloor \mu \rfloor = 2m + 1$ if $n \geq m^2 + m$.

The function f_n is decreasing on $(0, \sqrt{n})$ and increasing on $[\sqrt{n}, \infty)$, hence the minimum of f_n on $(0, \infty)$ is $f(\sqrt{n}) = 2\sqrt{n}$. It immediately follows that $\mu = \min\{f(m), f(m+1)\}$. Now, $f(m+1) - f(m) = m + 1 + \dfrac{n}{m+1} - m - \dfrac{n}{m} = 1 - \dfrac{n}{m(m+1)}$ and therefore $\mu = f(m)$ if $n < m(m+1)$ and $\mu = f(m+1)$ if $n \geq m(m+1)$. In the former case, we have $m \leq \dfrac{n}{m} < m + 1$, hence $2m \leq \mu = m + \dfrac{n}{m} < 2m + 1$ and so $\lfloor \mu \rfloor = 2m$. In the latter case, $(m+1)^2 > n \geq m(m+1)$ and $\mu = f(m+1) = m + 1 + \dfrac{n}{m+1}$ satisfies $2m + 1 \leq \mu < 2m + 2$ so that $\lfloor \mu \rfloor = 2m + 1$.

Solution 2 by Albert Stadler, Herrlierg, Switzerland

We denote by $[x]$ the integer part of x and claim that the integer part of the minimal value of $k + n/k, k \in N$, equals either $[2\sqrt{n}]$ or $[2\sqrt{n}] + 1$, and it equals $[2\sqrt{n}] + 1$ if and only if there is a natural number m such that $n = m(m+1)$.

The function $x \to x + n/x$ is decreasing for $x < \sqrt{n}$ and increasing for $x > \sqrt{n}$. The minimum at $\sqrt{n}$ equals $2\sqrt{n}$.

Therefore

$$2\sqrt{n} \leq \min_{k \in N}\left(k + \frac{n}{k}\right) = \min\left([\sqrt{n}] + \frac{n}{[\sqrt{n}]},\ [\sqrt{n}] + 1 + \frac{n}{[\sqrt{n}] + 1}\right).$$

The inequality

$$[\sqrt{n}] + 1 + \frac{n}{[\sqrt{n}] + 1} < 2[\sqrt{n}] + 2$$

is equivalent to $\sqrt{n} < [\sqrt{n}] + 1$ which is true. So

$$[2\sqrt{n}] \leq \left[\min_{k \in N}\left(k + \frac{n}{k}\right)\right] \leq 2\left[\sqrt{n}\right] + 1.$$

Clearly, $2[\sqrt{n}] + 1 - [2\sqrt{n}] \in \{0, 1\}$. So the integer part of $\min\limits_{k \in N}\left(k + \dfrac{n}{k}\right)$ equals either $[2\sqrt{n}]$ or $[2\sqrt{n}] + 1$. It remains to investigate for which n we have

$$[2\sqrt{n}] + 1 = \min\left([\sqrt{n} + \left[\frac{n}{[\sqrt{n}]}\right],\ [\sqrt{n}] + 1 + \left[\frac{n}{[\sqrt{n}] + 1}\right]\right). \qquad (*)$$

Clearly, if $n = m(m+1)$ above equation holds true, since $[\sqrt{n}] = m$, the right-hand side equals $2m + 1$ and the left-hand side equals

$$[2\sqrt{n}] + 1 = 2m + 1 + \left[2\sqrt{m(m+1)} - 2m\right] = 2m + 1 + \Bigg[\underbrace{\frac{2m}{\sqrt{m(m+1)} + m}}_{<1}\Bigg] = 2m + 1$$

as well. It remains to prove that if (*) holds true then $n = m(m+1)$ for some natural number m.

Let $m = [\sqrt{n}],\ r = n - m^2$. Then $0 \leq r \leq 2m$, and

$$[2\sqrt{n}]+1 = [2\sqrt{m^2+r}]+1 = 2m+1+[2\left(\sqrt{m^2+r}-m\right)] = \begin{cases} 2m+1, & 0 \le r \le m \\ 2m+2, & m+1 \le r \le 2m \end{cases}$$

$$[\sqrt{n}] + \left[\frac{n}{[\sqrt{n}]}\right] = m + \left[\frac{m^2+r}{m}\right] = 2m + \left[\frac{r}{m}\right] = \begin{cases} 2m & 0 \le r \le m-1 \\ 2m+1, & m \le r \le 2m-1 \\ 2m+2 & r = 2m \end{cases}$$

$$[\sqrt{n}]+1\left[\frac{n}{[\sqrt{n}]+1}\right] = m+1+\left[\frac{m^2+r}{m+1}\right] = 2m+\left[\frac{r+1}{m+1}\right] = \begin{cases} 2m & 0 \le r \le m-1 \\ 2m+1, & m \le r \le 2m \end{cases}$$

This shows that (*) can only hold true for $r = m$ which implies that $n = m^2 + m = m(m+1)$, as claimed.

Solution 3 by Kee-Wai Lau, Hong-Kong, China

We show that the integer part of the minimal value of $k + \dfrac{n}{k}$, $k \in N$ equals

$$\begin{cases} 2\lfloor\sqrt{n}\rfloor, & \lfloor\sqrt{n}\rfloor^2 \le n < \lfloor\sqrt{n}\rfloor\left(\lfloor\sqrt{n}\rfloor+1\right) \\ 2\lfloor\sqrt{n}\rfloor+1, & \lfloor\sqrt{n}\rfloor\left(\lfloor\sqrt{n}\rfloor+1\right) \le n < \left(\lfloor\sqrt{n}\rfloor+1\right)^2 \end{cases}$$

where $\lfloor t \rfloor$ is the greatest integer not exceeding t.

Suppose that $m^2 \le n < (m+1)^2$, where m is any positive integer.

For real x, the convex function $x + \dfrac{n}{x}$ attains its minimal value when $x = \sqrt{n}$.

Hence the minimal value of $k + \dfrac{n}{k}, k \in N$ equals

$$\min\left(m + \frac{n}{m}, m+1+\frac{n}{m+1}\right) = \begin{cases} m + \dfrac{n}{m}, & m^2 \le n < m(m+1) \\ m+1+\dfrac{n}{m+1}, & m(m+1) \le n < (m+1)^2. \end{cases}$$

Hence our claim.

Solution 4 by Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA

The integer part of the minimal value of $k + \frac{n}{k}, k \in N$ is given by

$$\begin{cases} \lceil n+1 \rceil & \text{if } n \le -1 \\ \lfloor n+1 \rfloor & \text{if } -1 < n < 1 \\ 2\left\lfloor\sqrt{n}\right\rfloor & \text{if } 1 \le n < \left\lfloor\sqrt{n}\right\rfloor^2 + \left\lfloor\sqrt{n}\right\rfloor \\ 2\left\lfloor\sqrt{n}\right\rfloor + 1 & \text{if } n > 1 \text{ and } \ge \left\lfloor\sqrt{n}\right\rfloor^2 + \left\lfloor\sqrt{n}\right\rfloor \end{cases}$$

Consider the function $f(x) = x + \frac{n}{x}$, where x is a positive real number. Then its derivative $f'(x) = 1 - \frac{n}{x^2}$ is positive for all $x \in N$ if $n < 1$, in which case the minimum value of $k + \frac{n}{k}$ is $1 + n$. Recall that if $n + 1 < 0$, then the integer part of $n + 1$ is $\lceil n+1 \rceil$. On the other hand, if n is a positive real number, then $f'(x) < 0$ for $0 < x < \sqrt{n}$ and $f'(x) > 0$ for $x > \sqrt{n}$, so that $f(\sqrt{n}) = 2\sqrt{n}$ is the minimum value of $f(x)$ over the continuous interval $(0,\infty)$. If $n=a^2$, where $a\in N$, then the minimum value of $k+\frac{n}{k}$, where $k\in N$, is $f(a)=2\sqrt{n}=2a$, which is an integer. If $n>1$ and n is not a perfect square, then the minimum value of $k+\frac{n}{k}$, where $k\in N$, is either

$$f(\lfloor\sqrt{n}\rfloor)=\lfloor\sqrt{n}\rfloor+\frac{n}{\lfloor\sqrt{n}\rfloor}$$

or

$$f(\lceil\sqrt{n}\rceil)=\lceil\sqrt{n}\rceil+\frac{n}{\lceil\sqrt{n}\rceil}=\lfloor\sqrt{n}\rfloor+1+\frac{n}{\lfloor\sqrt{n}\rfloor+1}.$$

If $n>1$, notice that $\lfloor\sqrt{n}\rfloor\le\sqrt{n}$, so that $\lfloor\sqrt{n}\rfloor^2\le n$ and $\frac{n}{\lfloor\sqrt{n}\rfloor}\ge\lfloor\sqrt{n}\rfloor$, so that

$$f(\lfloor\sqrt{n}\rfloor)=\lfloor\sqrt{n}\rfloor+\frac{n}{\lfloor\sqrt{n}\rfloor}\ge 2\lfloor\sqrt{n}\rfloor.$$

In addition, $(\lfloor\sqrt{n}\rfloor+1)(\lfloor\sqrt{n}\rfloor-1)=\lfloor\sqrt{n}\rfloor^2-1\le n-1<n$, so that $\frac{n}{\lfloor\sqrt{n}\rfloor+1}>\lfloor\sqrt{n}\rfloor-1$, and

$$f(\lceil\sqrt{n}\rceil)=\lfloor\sqrt{n}\rfloor+1+\frac{n}{\lfloor\sqrt{n}\rfloor+1}>\lfloor\sqrt{n}\rfloor+1+\lfloor\sqrt{n}\rfloor-1=2\lfloor\sqrt{n}\rfloor.$$

Since $\sqrt{n}<\lfloor\sqrt{n}\rfloor+1$, then $n<(\lfloor\sqrt{n}\rfloor+1)^2$ and $\frac{n}{\lfloor\sqrt{n}\rfloor+1}<\lfloor\sqrt{n}\rfloor+1$ so

$$f(\lceil\sqrt{n}\rceil)=\lfloor\sqrt{n}\rfloor+1+\frac{n}{\lfloor\sqrt{n}\rfloor+1}<2\lfloor\sqrt{n}\rfloor+2.$$

Thus, for $n>1$, the integer part of the minimum value of $f(k)=k+\frac{n}{k}$ is either $2\lfloor\sqrt{n}\rfloor$ or $2\lfloor\sqrt{n}\rfloor+1$. We consider two cases, depending on whether $n<\lfloor\sqrt{n}\rfloor^2+\lfloor\sqrt{n}\rfloor$. First, notice that

$$f(\lfloor\sqrt{n}\rfloor)=\lfloor\sqrt{n}\rfloor+\frac{n}{\lfloor\sqrt{n}\rfloor}=2\lfloor\sqrt{n}\rfloor+\frac{n-\lfloor\sqrt{n}\rfloor^2}{\lfloor\sqrt{n}\rfloor}$$

and

$$f(\lceil\sqrt{n}\rceil)=\lfloor\sqrt{n}\rfloor+1+\frac{n}{\lfloor\sqrt{n}\rfloor+1}=2\left(\lfloor\sqrt{n}\rfloor+1\right)-\frac{(\lfloor\sqrt{n}\rfloor+1)^2-n}{\lfloor\sqrt{n}\rfloor+1}.$$

Case 1: $n<\lfloor\sqrt{n}\rfloor^2+\lfloor\sqrt{n}\rfloor$. Then $\frac{n}{\lfloor\sqrt{n}\rfloor}<\lfloor\sqrt{n}\rfloor+1$ and

$$f(\lfloor\sqrt{n}\rfloor)=\lfloor\sqrt{n}\rfloor+\frac{n}{\lfloor\sqrt{n}\rfloor}<2\lfloor\sqrt{n}\rfloor+1.$$

In addition,

$$\left(\lfloor\sqrt{n}\rfloor+1\right)^2-n=\lfloor\sqrt{n}\rfloor^2+\lfloor\sqrt{n}\rfloor-n+\lfloor\sqrt{n}\rfloor+1>\lfloor\sqrt{n}\rfloor+1,$$

so that

$$f(\lceil\sqrt{n}\rceil)=2\left(\lfloor\sqrt{n}\rfloor+1\right)-\frac{(\lfloor\sqrt{n}\rfloor+1)^2-n}{\lfloor\sqrt{n}\rfloor+1}<2\left(\lfloor\sqrt{n}\rfloor+1\right)-1=2\lfloor\sqrt{n}\rfloor+1.$$

Thus, in Case 1, the integer part of the minimal value of $f(k)$ is $2\lfloor\sqrt{n}\rfloor$.
Case 2: $n\ge\lfloor\sqrt{n}\rfloor^2+\lfloor\sqrt{n}\rfloor$. Then

$$f(\lfloor\sqrt{n}\rfloor)=2\lfloor\sqrt{n}\rfloor+\frac{n-\lfloor\sqrt{n}\rfloor^2}{\lfloor\sqrt{n}\rfloor}\ge 2\lfloor\sqrt{n}\rfloor+1.$$

In addition,

$$\left(\lfloor\sqrt{n}\rfloor+1\right)^2-n=\lfloor\sqrt{n}\rfloor^2+\lfloor\sqrt{n}\rfloor-n+\lfloor\sqrt{n}\rfloor+1\le\lfloor\sqrt{n}\rfloor+1,$$

so that

$$f(\lceil\sqrt{n}\rceil)=2\left(\lfloor\sqrt{n}\rfloor+1\right)-\frac{(\lfloor\sqrt{n}\rfloor+1)^2-n}{\lfloor\sqrt{n}\rfloor+1}\ge 2\left(\lfloor\sqrt{n}\rfloor+1\right)-1=2\lfloor\sqrt{n}\rfloor+1.$$

Thus, in Case 2, the integer part of the minimal value of $f(k)$ is $2\lfloor\sqrt{n}\rfloor+1$.

Solution 5 by Brian D. Beasley, Presbyterian College, Clinton, SC

For each real number n, we define $f(n)=\lfloor\min\{k+n/k : k\in N\}\rfloor$ and note that f is a non-decreasing function on R. If $n\le 1$, then the minimum value of $k+n/k$ for $k\in N$ occurs when $k=1$, so $f(n)=\lfloor 1+n\rfloor$. If $n>1$, then there is a unique positive integer m with either

$$m^2\le n<m(m+1)\quad\text{or}\quad m(m+1)\le n<(m+1)^2.$$

We observe that f increases by 1 only at each $n=m^2$ and at each $n=m(m+1)$: Let $\varepsilon\in(0,1)$. Then $f(m^2)=2m$, while $f(m^2-\varepsilon)=2m-1$ by taking $k=m$. Similarly, $f(m(m+1))=2m+1$, while $f(m(m+1)-\varepsilon)=2m$ by taking $k=m$ or $k=m+1$. Hence we conclude that if $m^2\le n<m(m+1)$, then $f(n)=2m$, while if $m(m+1)\le n<(m+1)^2$, then $f(n)=2m+1$.

Addendum. It is interesting to note that when n is a positive integer, then $f(n)=\lfloor 2\sqrt{n}\rfloor$ unless $n=m(m+1)$, in which case $f(n)=\lfloor 2\sqrt{n}\rfloor+1$.

Solution 6 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

For an integer n, let $M(n)=\left[\min\left\{k+\dfrac{n}{k}, k\in N\right\}\right]$,
where [.] is the greatest integer function. We shall see that

$$M(n)=\begin{cases} n+1,\ \text{if } n\le 0;\\ [2\sqrt{n}]+1, \text{if } n\ge 1 \text{ has the form } m(m+1);\\ [2\sqrt{n}]\,,\ \text{otherwise}\end{cases}.$$

Note that in the second case, $M(n)=M\left(m^2+m\right)=2m{+}1$.

By calculus, we know that the function $f_n(k)=k+\dfrac{n}{k}$, with k considered as a continuous (positive) variable, achieves an absolute minimum of $2\sqrt{n}$ at the sole critical point $\sqrt{n}$.

Thus, when we restrict k to integer values, the minimum will be close to $2\sqrt{n}$ and this restricted minimum must occur near $\sqrt{n}$. That is, it must occur at $k=[\sqrt{n}]$ or at $k=[\sqrt{n}]+1$.

Let us validate our claim. For $n\le 0$, $\mathrm{f}_n(1)=1+n$ while $f_n(k)=k+\dfrac{n}{k}$ for $k>1$.

Thus the minimal value is the integer $n+1$, which therefore is $M(n)$.

Now suppose that n is positive and trapped between given consecutive squares: $m^2\le n<(m+1)^2$.

Thus, $m \leq \sqrt{n} < m+1$, so $m = [\sqrt{n}]$.

In the nice case where n is a square, $m^2 = n$, then choosing k to be n yields the calculus-predicted absolute minimum: $f_n(m) = m + \dfrac{m^2}{m} = 2m = 2\sqrt{n} = [2\sqrt{n}] = M(n)$.

In the special case $n = m(m+1) = m^2 + m$, we find more nice behavior. The two candidates for the occurrence of the minimum are at $k = [\sqrt{n}]$ or $k = [\sqrt{n}] + 1$. But these are just m and $m+1$, and

$f_n(m) == m + \dfrac{m(m+1)}{m} = 2m+1, \text{and } f_n(m+1) = m+1+\dfrac{m(m+1)}{m+1} = 2m+1.$

Therefore $M(n) = 2m+1$, which is $[2\sqrt{n}]+1$.

Finally, consider the case that $m^2 < n < (m+1)^2$ and $n \neq m(m+1)$.

We still have $m \leq \sqrt{n} < m+1$, so $m = [\sqrt{n}]$. The two candidates for the location of our minimum value: at $k = m$ or $k = m+1$. Some algebra shows that

$$f_n(m) = m + \frac{n}{m} < f_n(m+1) = m+1+\frac{n}{m+1} \iff n < m(m+1).$$

So the location of n in the interval $\left(m^2, (m+1)^2\right)$ determines the appropriate choice for k. But for each choice, the minimum value turns out to be $[2\sqrt{n}]$.

We present the (ticky) details verifying that the first choice behaves as claimed: $n < m(m+1)$ using $k = m$, so that $f_n(k) = m + \dfrac{n}{m}$.

Because $m^2 < n < m(m+1)$, we have $m < \dfrac{n}{m} < m+1$, so $\left[\dfrac{n}{m}\right] = m$.

Therefore $[f_n(k)] = \left[n + \dfrac{n}{m}\right] = m + m = 2m = M(n)$.

Moreover, $[2\sqrt{n}] = 2m$ also. This is true because (1) $2m < 2\sqrt{n}$; and if we had $2m+1 < 2\sqrt{n}$, we would conclude by squaring that $4m^2+4m+1 < 4n < 4m^2+4m$, which is a contradiction by our choice for n.

Therefore, $M(n) = 2m = [2\sqrt{n}]$.

The argument for $k = m+1$ is similar.

The proof of our formula for $M(n)$ is complete.

Comment (by authors): This interesting problem has connections to three classic problems.

- (1) The ancient Babylonian method for computing the square root of n: make a guess ,k. Compute n/k, then average the result with k, producing a better approximation to the desired root. Repeat as long as you want to the method converges to n.
- (2) But of course, this method turns out to be Newton's method applied to the function $f(x) = x - n^2$.
- (3) The favorite Calculus I example, done in class to help the students see the power of calculus and understand graphs: "What does the graph of $f(x) = x + \dfrac{n}{x}$ look like?

We have two terms competing; for positive x close to zero, the $\dfrac{n}{x}$ term dominates and the graph climbs to infinity; for big positive x, the x term wins and the graph also climbs to infinity. $f(x)$ is continuous and always positive, so the graph must "min out" somewhere.